There are several families of subgroups that every group has that depend on its structure.

Centralizers and Normalizers

Let G be a group and $A \subseteq G$ a subset (not necessarily a subgroup.

Def: The <u>centralizer</u> of A, denoted $C_G(A)$, is the set of elements of G that commute w/ all elements of A.

i.e. $C_G(A) := \{ g \in G \mid gag^{-1} = a \ \forall \ a \in A \}$.

(Note that $gag^{-1} = a \iff ga = ag$.)

Claim: $C_G(A) \leq G$.

Pf: $1 \in C_G(A)$, so it's nonempty.

If $x, y \in C_G(A)$, we want to show $xy^{-1} \in C_G(A)$ as well:

Let $a \in A$. Then $ya = ay \Rightarrow a = y^{-1}ay$

$$\Rightarrow xax^{-1} = x(y^{-1}ay)x^{-1}$$
$$\Rightarrow a = (xy^{-1})a(yx^{-1}) = (xy^{-1})a(xy^{-1})^{-1}$$

Thus, $xy^{-1} \in C_G(A)$, so it's a subgroup. $\square$

Ex: 1.) $C_G(1) = G$, since everything commutes w/ the identity.

2.) From the homework, if $n = 2k$,

then $C_{D_{2n}}(r^k) = D_{2n}$ and $C_{D_{2n}}(D_{2n}) = \{1, r^k\}$.

The subgroup $C_G(G)$ is the set of elements that commute with every element of G, and is denoted $Z(G)$. It is called the center of G.

Note that $Z(G) = G \Longleftrightarrow G$ is abelian.

Ex: From the homework, if $n \geq 3$ then

- If $n = 2k$, then $Z(D_{2n}) = \{1, r^k\}$
- If n is odd, $Z(D_{2n}) = 1$.

Def: Define $gAg^{-1} = \{gag^{-1} \mid a \in A\}$. The normalizer of A in G is the set $N_G(A) = \{g \in G \mid gAg^{-1} = A\}$

(It will be clear soon why it's called the normalizer and why it's useful.)

The normalizer is also a subgroup of G (proof is nearly identical to the one for the centralizer).

Notice: Being in the normalizer of a set is weaker than being in its centralizer. That is,

If $g \in C_G(A)$ then $gag^{-1} = a \quad \forall \quad a \in A$ so

$gAg^{-1} = A$. Thus $g \in N_G(A) \implies C_G(A) \leq N_G(A)$.

Ex: Consider $S_3 = \{1, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}$

Let $A = \{1, (1\,2)\}$.

What is $C_{S_3}(A)$? Well, by Lagrange's Theorem (see HW3), $|C_{S_3}(A)| \mid 6$. But also $2 \mid |C_{S_3}(A)|$ since $A \leq C_{S_3}(A)$.

So $|C_{S_3}(A)| = 2$ or 6, but $(1\,2)(1\,3) \neq (1\,3)(1\,2)$ (They send 1 different places), so $C_{S_3}(A) = A$.

For the normalizer of A, $\sigma \in N_{S_3}(A)$ iff

$$\{\sigma(1\,2)\sigma^{-1}, \underset{\substack{\parallel \\ 1}}{\sigma 1 \sigma^{-1}}\} = \{(1\,2), 1\}.$$

$\iff \sigma(1\,2)\sigma^{-1} = (1\,2)$. But this would imply $\sigma \in C_{S_3}(A)$, so $N_{S_3}(A) = C_{S_3}(A)$.

Ex: Consider D_8 again. Let $A = \{1, r, r^2, r^3\}$.

By Lagrange's Theorem, since $sr \neq rs$, $|C_{D_8}(A)| = 4$,

so $C_{D_8}(A) = A$.

However, this means $N_{D_8}(A) = C_{D_8}(A)$ or D_8.

$s r^i s = s^2 r^{-i} = r^{-i} \in A$, so $s \in N_{D_8}(A)$, so $N_{D_8}(A) = D_8$.

Stabilizers of group actions

Notice that the centralizer and normalizer are subgroups determined by G acting on subsets of itself by "conjugation" (we'll come back to this shortly).

We can generalize these subgroups to arbitrary group actions.

Let G be a group acting on a set A.

Def: Let $a \in A$. The stabilizer of a in G is the set

$$G_a = \{ g \in G \mid g \cdot a = a \}$$

Claim: $G_a \leq G$.

Pf: $1 \cdot a = a$, so $G_a \neq \emptyset$. If $x, y \in G_s$, then

$x^{-1} \cdot a = x^{-1}(x \cdot a) = 1 \cdot a = a$, so $x^{-1} \in G_s$.

$(xy) \cdot a = a$. So $G_a \leq G$. $\square$

Ex: Let $G = S_4$. Consider the action of G on $\{1,2,3,4\}$ given by $\sigma \cdot a = \sigma(a)$. What is G_4?

G_4 consists of cycle decompositions that don't contain 4.
i.e. $G_4 = \{1, (12), (13), (23), (123), (132)\}$. Note that this is natural isomorphic to S_3.

Def: The <u>kernel</u> of a group action is $\{g \in G \,/\, g \cdot a = a \;\forall\, a \in A\}$

Note that this is the set of all elements of g that act as the identity on A.

i.e. if $\sigma_g : A \to A$ is defined (as before) as $\sigma_g(a) = g \cdot a$,

then the kernel is the set of g s.t. σ_g = identity.

In other words, the kernel of the group action is equal to the kernel of the map we defined

$$G \to S_A \text{ by } g \mapsto \sigma_g.$$

So the kernel of the group action is equal to the kernel of a group homomorphism, so it must be a subgroup.

Just as before, the action is faithful $\iff$ its kernel $= 1$.

Conjugation as a group action

Let G be a group. Let $S = \mathcal{P}(G)$ = the set of all subsets of G.

Let G act on S by conjugation. That is, if $A \in S$, define $g \cdot A = gAg^{-1} = \{h \in G \mid h = gag^{-1} \text{ for some } a \in A\} \in S$.

Claim: This is in fact a group action.

Pf: If $A \in S$, then $1 \cdot A = A$.

For $g, h \in G$, we have

$$(gh) \cdot A = (gh)A(gh)^{-1} = (gh)A(h^{-1}g^{-1}) = \{k \mid k = ghah^{-1}g^{-1}, \text{ some } a \in A\}$$
$$= g(hAh^{-1})g^{-1} = g \cdot (hAh^{-1}) = g \cdot (h \cdot A). \quad \square$$

Now, for any subset $A \subseteq G$, we have $A \in \mathcal{P}(G)$.
So $G_A = \{g \mid gAg^{-1} = A\} = N_G(A)$. That is,

The normalizer of A is equal to the stabilizer of A under the action of conjugation.

We can also act on individual elements by conjugation:

Claim: G acts on itself by $g \cdot a = gag^{-1}$.

Pf: $1 \cdot a = a$, and if $g, h \in G$,

$$(gh)\cdot a = (gh)a(gh)^{-1} = ghah^{-1}g^{-1} = g(h\cdot a)g^{-1} = g\cdot(h\cdot a). \square$$

Note that the kernel of this action is exactly

$$\{g \in G \mid gag^{-1} = a \ \forall\, a \in G\} = Z(G).$$

Moreover, for any subset $A \subseteq G$, $N_G(A)$ acts on A by conjugation (by construction).

Then the kernel of this action is

$$\{g \in N_G(A) \mid gag^{-1} = a \ \ \forall\, a \in A\} = N_G(A) \cap C_G(A)$$

But $C_G(A) \leq N_G(A)$, so

The kernel of the action of $N_G(A)$ on A by conjugation is equal to $C_G(A)$.